

GRÖBNER-SHIRSHOV BASES: SOME NEW RESULTS*

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Abstract: In this survey article, we report some new results of Gröbner-Shirshov bases, including new Composition-Diamond lemmas, applications of some known Composition-Diamond lemmas and content of some expository papers.

Key words: Composition-Diamond lemma; Group; HNN-extension; Schreier extension; Dialgebra; Lie algebra; Module; Chinese monoid.

1 Introduction

In this survey, we report the activities of the first author who has been staying in the South China Normal University, at Guangzhou, in Spring 2006, Spring 2007 and November-December 2007. With the participation of the second author, we are running an algebra seminar with 12 students (most of them are master degree students) and some young teachers at 5 times per week, two hours each session. The subjects of this seminar include from Combinatorial group theory, Free Lie algebras, Semi-simple Lie algebras to Non-associative algebras, Conformal algebras, Quantum groups, Semigroups and Dialgebras with emphasizing in Gröbner-Shirshov bases.

The second author visited Sobolev Institute of Mathematics at Novosibirsk as a visiting professor in July-October, 2006.

As the result of all these activities, more than 10 papers have been prepared. We now give a brief survey for some of the papers. We also mention some papers which were done

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during the first author's visiting to Prof. K. P. Shum, the Chinese University of Hong Kong, 2003-2005.

We divide these papers into three blocks:

- (i) *New Composition-Diamond (CD-) lemmas,*
- (ii) *Applications of known CD-lemmas,*
- (iii) *Expository papers.*

We first explain what it means of “CD-Lemma” for a class (variety or category) \mathcal{M} of linear Ω -algebras over a field k (here Ω is a set of linear operations on \mathcal{M}) with free objects.

\mathcal{M} -CD-Lemma Let \mathcal{M} be a class of (in general, non-associative) Ω -algebras, $Free_{\mathcal{M}}(X)$ a free Ω -algebra in \mathcal{M} generated by X with a linear base consisting of “normal (non-associative Ω -) words” $[u]$, $S \subset Free_{\mathcal{M}}(X)$ a subset and $<$ a monomial well order on normal words. Let S be a Gröbner-Shirshov basis (this means that any “composition” of elements in S is “trivial”). Then

- (a) If $f \in Ideal(S)$, then $[\bar{f}] = [a\bar{s}b]$, where $[\bar{f}]$ is the “leading monomial” of f and $[asb]$ is a “normal s -word”, $s \in S$.
- (b) $Irr(S) = \{[u] \mid [u] \neq [a\bar{s}b], s \in S, [asb] \text{ is a normal } s\text{-word}\}$ is a linear basis of the algebra $\mathcal{M}(X|S)$ with defining relations S .

In many cases, each of conditions (a) and (b) is equivalent to the condition that S is a Gröbner-Shirshov basis in $Free_{\mathcal{M}}(X)$. But in some of our “new CD-Lemmas”, this is not the case. Here, \mathcal{M} may be as follows:

–Associative algebras ([56], [7], [3]).

A free associative algebra is $k\langle X \rangle$, the algebra of non-commutative polynomials; normal words are words on X ; the A. I. Shirshov's composition $(f, g)_w$ is equal to $fb - ag$, if $w = \bar{f}b = a\bar{g}$, $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$, or $f - agb$, if $w = \bar{f} = a\bar{g}b$, where $a, b \in X^*$ and X^* the free monoid generated by X ; a polynomial h is called trivial $mod(S)$ if it goes to 0 by using the Eliminations of Leading Words (ELW) of S (see below an equivalent definition).

–Lie algebras ([56]).

A free Lie algebra is $Lie(X)$, the algebra of Lie polynomials in $k\langle X \rangle$ (this theorem was proved by W. Magnus and E. Witt); by the normal words we mean the non-associative Lyndon-Shirshov words $[u]$ on X ; the leading word \bar{f} of a Lie polynomial f is the same as the associative polynomial; A. I. Shirshov's composition $[f, g]_w$ of two Lie polynomials is its associative composition with some extra bracketing defined in [53]; a normal s -word for $s \in Lie(X)$ has the form $[asb]$ with extra bracketing as before.

–Commutative algebras ([17], [18]).

A free commutative algebra is $k[X]$, the algebra of polynomials on X over a field k ; normal words are monomials; the composition $S(-, -)$ is the operation of taking

the B. Buchberger's S -polynomial: $S(f, g) = fb - ag$ for any polynomials f, g , where $w = \bar{f}b = a\bar{g} = \text{l.c.m}(\bar{f}, \bar{g})$ and $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$.

–(Commutative, anti-commutative) non-associative algebras ([55]).

There is only composition of inclusion in the cases.

–Lie superalgebras ([45]).

The Composition-Diamond lemma for Lie superalgebras is known and proved.

–Grassmann algebras ([57]).

There is new composition of multiplication by a monomial.

–Supercommutative associative superalgebras ([45], [46]).

There is new composition of multiplication by a monomial.

–Conformal associative algebras $(C, (n), n \geq 0, D)$ ([12]).

There are 6 types of compositions including inclusion, intersection, D -inclusion, D -intersection, left (right) multiplication by a generator. The condition (a) ((b)) in the CD-lemma is not equivalent to the condition that S is a Gröbner-Shirshov basis.

–Modules ([36], [29]).

2 CD-lemma for associative algebras

In this section, we cite some concepts and results from the literature which are related to the Gröbner-Shirshov bases for the associative algebras.

Definition 2.1 ([56], see also [6], [7]) *Let f and g be two monic polynomials in $k\langle X \rangle$ and $<$ a well order on X^* . Then, there are two kinds of compositions:*

(i) *If w is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$, then the polynomial $(f, g)_w = fb - ag$ is called the intersection composition of f and g with respect to w .*

(ii) *If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the inclusion composition of f and g with respect to w .*

Definition 2.2 ([6], [7], [56]) *Let $S \subset k\langle X \rangle$ with each $s \in S$ monic. Then the composition $(f, g)_w$ is called trivial modulo (S, w) if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$. If this is the case, then we write $(f, g)_w \equiv 0 \pmod{(S, w)}$.*

Definition 2.3 ([6], [7], [56]) *We call the set S with respect to the well order “ $<$ ” a Gröbner-Shirshov set (basis) in $k\langle X \rangle$ if any composition of polynomials in S is trivial modulo S .*

If a subset S of $k\langle X \rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S , and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis S^{comp} . Such a process is called the Shirshov algorithm. It is an infinite algorithm as well as Kruth-Bendix algorithm (see [38]).

A well order $>$ on X^* is monomial if it is compatible with the multiplication of words, that is, for $u, v \in X^*$, we have

$$u > v \Rightarrow w_1 u w_2 > w_1 v w_2, \text{ for all } w_1, w_2 \in X^*.$$

A standard example of monomial order on X^* is the deg-lex order to compare two words first by degree and then lexicographically, where X is a linearly ordered set.

The following lemma was proved by Shirshov [56] for the free Lie algebras (with deg-lex ordering) in 1962 (see also Bokut [6]). In 1976, Bokut [7] specialized the approach of Shirshov to associative algebras, see also Bergman [3]. For commutative polynomials, this lemma is known as the Buchberger's Theorem in [17] and [18].

Lemma 2.4 (*Composition-Diamond Lemma*) *Let k be a field, $A = k\langle X | S \rangle = k\langle X \rangle / Id(S)$ and $<$ a monomial order on X^* , where $Id(S)$ is the ideal of $k\langle X \rangle$ generated by S . Then the following statements are equivalent:*

- (i) S is a Gröbner-Shirshov basis in $k\langle X \rangle$.
- (ii) $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.
- (iii) $Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$ is a basis of the algebra $A = k\langle X | S \rangle$.

3 New CD-lemmas

3.1 CD-Lemma and HNN-extensions

Y. Q. Chen and C. Y. Zhong in [27] give a version of CD-lemma in which the order may not be monomial. A Gröbner-Shirshov basis for HNN extensions of groups is obtained by using the new CD-lemma. This is the first paper to give a Gröbner-Shirshov basis by using a non-monomial order.

Theorem 3.1 ([27]) *Let $S \subseteq k\langle X \rangle$ and “ $<$ ” a well order on X^* such that*

- (i) $\overline{a\bar{s}b} = a\bar{s}b$ for any $a, b \in X^*, s \in S$;
- (ii) for each composition $(s_1, s_2)_w$ in S , there exists a presentation

$$(s_1, s_2)_w = \sum_i \alpha_i a_i t_i b_i, \quad a_i \bar{t}_i b_i < w, \quad \text{where } t_i \in S, a_i, b_i \in X^*, \alpha_i \in k$$

such that for any $c, d \in X^$, we have $ca_i \bar{t}_i b_i d < cwd$.*

Then, the following statements hold.

(i) S is a Gröbner-Shirshov basis in $k\langle X \rangle$.

(ii) For any $f \in k\langle X \rangle$, $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$, $a, b \in X^*$.

(iii) The set

$$Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$$

is a linear basis of the algebra $k\langle X | S \rangle$.

We call the order satisfying the conditions in Theorem 3.1 an S -weak monomial order.

Let $G = gp\langle H, t | t^{-1}at = \varphi(a), a \in A \rangle$ be an HNN-extension of a group H , where A is a subgroup of H and φ a group isomorphism. By using Theorem 3.1, it is proved in [[27]] that there exists an explicit Gröbner-Shirshov basis S of G relative to some explicit S -weak monomial order such that the set $Irr(S)$ of S -irreducible words coincides with the set of normal forms in the Normal Form Theorem for HNN-extensions (see [[4]] and [[43]]).

3.2 Dialgebras

In this section, we report some recent results of L. A. Bokut, Y. Q. Chen and C. H. Liu [10].

Let $D(X)$ be a free dialgebra (J.-L. Loday, 1995, [40]), where multiplications “ \vdash ”, “ \dashv ” are both associative and for any $a, b, c \in D(X)$,

$$\begin{aligned} a \dashv (b \vdash c) &= a \dashv b \dashv c, \\ (a \dashv b) \vdash c &= a \vdash b \vdash c, \\ a \vdash (b \dashv c) &= (a \vdash b) \dashv c. \end{aligned}$$

A linear basis of $D(X)$ consists of normal diwords

$$[u] = x_{-m} \vdash \cdots \vdash x_0 \dashv \cdots \dashv x_k = x_{-m} \cdots x_0 \cdots x_k,$$

where $x_i \in X$, $m, k \geq 0$, x_0 is the center of $[u]$ (see J.-L. Loday, 1995, [40]). We define the deg-lex order $[u] < [v]$, by using the lex-order of the weight $wt[u] = (k + m + 1, m, x_{-m}, \dots, x_k)$.

Now, for $f, g \in S$, we define the compositions of inclusion, intersection and left(right) multiplication by a letter.

We call the set S a Gröbner-Shirshov set (basis) in $D(X)$ if any composition of polynomials in S is trivial modulo S (and $[w]$).

Theorem 3.2 ([10] *CD-Lemma for dialgebras*) *Let $S \subset D(X)$ be a monic set and the order $<$ as before. Then (i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv), where*

(i) S is a Gröbner-Shirshov basis in $D(X)$.

(ii) For any $f \in D(X)$, $f \in Id(S) \Rightarrow [\bar{f}] = [a[\bar{s}]b]$ for some $s \in S$, $a, b \in [X^*]$ and $[asb]$ a normal S -diword.

(iii) The set $\text{Irr}(S) = \{u \in [X^*] \mid u \neq [a[\bar{s}]b], s \in S, a, b \in [X^*], [asb] \text{ is normal } S\text{-diword}\}$ is a linear basis of the dialgebra $D(X|S)$.

(iv) Each composition is trivial modulo S .

As an application of the above theorem, we obtain a Gröbner-Shirshov basis for the universal enveloping algebra of a Leibniz algebra. It is the PBW-Theorem for the Leibniz algebras. This is the third proof of the theorem after M. Aymon and P. P. Grival (2003) in [1], and P. Kolesnikov (2007) in [39].

Recall that a Leibniz algebra L is a non-associative algebra with a multiplication $[xy] \in L$ such that $[[xy]z] - [[xz]y] - [[yz]x] = 0$ (see [40]). For any dialgebra (D, \dashv, \vdash) , the linear space D with the multiplication $[xy] = x \dashv y - y \vdash x$ is a Leibniz algebra. For any Leibniz algebra $L = \text{Lei}(\{e_i\}_I \mid [e_i e_j] = \sum_k \alpha_{ij}^k e_k, i, j \in I)$, one can define the universal enveloping D -algebra $U(L) = D(\{e_i\}_I \mid e_i \dashv e_j - e_j \vdash e_i = \sum_k \alpha_{ij}^k e_k, i, j \in I)$, where $\{e_i\}_I$ is a basis of L .

Theorem 3.3 ([10]) *Let \mathcal{L} be a Leibniz algebra over a field k with the product $\{\cdot, \cdot\}$. Let \mathcal{L}_0 be the subspace of \mathcal{L} generated by the set $\{\{a, a\}, \{a, b\} + \{b, a\} \mid a, b \in \mathcal{L}\}$. Let $\{x_i \mid i \in I_0\}$ be a basis of \mathcal{L}_0 and $X = \{x_i \mid i \in I\}$ a linearly ordered basis of \mathcal{L} such that $I_0 \subseteq I$. Let $(D(X), \dashv, \vdash)$ be the free dialgebra and the order $<$ on $[X^*]$ as before. Let S be the set which consists of the following polynomials:*

1. $f_{ji} = x_j \vdash x_i - x_i \dashv x_j + \{x_i, x_j\} \quad (i, j \in I)$
2. $f_{j\vdash t} = x_j \vdash x_i \vdash x_t - x_i \vdash x_j \vdash x_t + \{x_i, x_j\} \vdash x_t \quad (i, j, t \in I, j > i)$
3. $h_{i_0 \vdash t} = x_{i_0} \vdash x_t \quad (i_0 \in I_0, t \in I)$
4. $f_{t \dashv j i} = x_t \dashv x_j \dashv x_i - x_t \dashv x_i \dashv x_j + x_t \dashv \{x_i, x_j\} \quad (i, j, t \in I, j > i)$
5. $h_{t \dashv i_0} = x_t \dashv x_{i_0} \quad (i_0 \in I_0, t \in I).$

Then

(i) S is a Gröbner-Shirshov basis in $(D(X))$.

(ii) The set

$$\{x_j \dashv x_{i_1} \dashv \cdots \dashv x_{i_k} \mid j \in I, i_p \in I - I_0, 1 \leq p \leq k, i_1 \leq \cdots \leq i_k, k \geq 0\}$$

is a linear basis of the universal enveloping algebra $U(\mathcal{L}) = D(X|S)$. In particular, \mathcal{L} can be embedded into $U(\mathcal{L})$.

3.3 Free Γ -algebras $k\langle X; \Gamma \rangle$

In this section, we summary the results given by L. A. Bokut and K. P. Shum [14].

Let X be a set, Γ a group, $\Gamma(x)$, $\Gamma'(x)$ isomorphic subgroups, $x \in X$. Then the algebra $k\langle X; \Gamma \rangle$ with defining relations

$$\gamma x = x \gamma' \quad (\gamma \in \Gamma(x), \gamma' \in \Gamma'(x), x \in X), \quad \gamma \delta = \mu \quad (\gamma, \delta, \mu \in \Gamma)$$

is called *free Γ -algebra*.

A linear basis of $k\langle X; \Gamma \rangle$ consists of Γ -words

$$u = \gamma_0 x_{i_1} \gamma_1 \cdots x_{i_k} \gamma_k, \quad x_i \in X, \gamma_i \in \Gamma, k \geq 0,$$

which are equivalent under transformations $\gamma x \rightarrow x \gamma'$ above.

We input a quasi-order on Γ -words:

$$u \leq v \Leftrightarrow [u] \leq [v],$$

where $[u] = x_{i_1} \cdots x_{i_k}$ is the projection of u , and $[u] \leq [v]$ a monomial order on X^* .

A Γ -polynomial f may have several leading monomials of \bar{f} . We call f a *strong* polynomial if \bar{f} is unique. We define compositions of inclusion and intersection of two strong Γ -polynomials, and a strong Γ -Gröbner-Shirshov basis. The later is a set of strong Γ -polynomials that is closed under compositions.

Theorem 3.4 *Let $k\langle X; \Gamma \rangle$ be a free strong Γ -algebra, $S \subset k\langle X; \Gamma \rangle$ a strong Γ -Gröbner-Shirshov basis. Then*

- (a) *If $f \in Id(S)$, then $\bar{f} = a\bar{s}b$, where \bar{f} is a leading monomial of f , $s \in S$, a, b Γ -words.*
- (b) *$Irr(S) = \{u \neq a\bar{s}b | s \in S, a, b \text{ are } \Gamma\text{-words}\}$ is a linear basis of $k\langle X; \Gamma | S \rangle$.*

There are many examples of Γ -algebras with strong Γ -Gröbner-Shirshov bases.

(a) Group algebras of universal groups $G(R^*)$ of multiplicative semigroups R^* of some rings R .

Let $R = \overline{k(S)}$, where $S = sgp\langle X | w_i h_i = u_i f_i, w_i, h_i, u_i, f_i \in X \rangle$, k a field, $k(S)$ the semigroup algebra, $\overline{k(S)}$ the algebra of formal series over S . In particular, if S is a free semigroup, then $\overline{k(S)} = \overline{k\langle X \rangle}$ is the Magnus algebra of formal series over X .

These examples are from Bokut's solution to the Malcev embedding problem: There exists a semigroup S such that $k(S)^* \subset G$ (the multiplicative semigroup of $k(S)$ is embeddable into a group), but $k(S) \not\subset D$ ($k(S)$ is not embeddable into any division ring) (see [5]).

(b) Group algebras $k(G)$ for Tits systems (G, B, N, S) (see [16]). Here G has strong Γ -Gröbner-Shirshov basis, where $\Gamma = B$ and Γ -normal form is the Bruhat normal form.

3.4 Tensor product of free algebras

In A. A. Mikhalev and A. A. Zolotykh [47], a CD-lemma for the algebra $k[X] \otimes k\langle Y \rangle$ was found, where $k[X]$ is a polynomial algebra generated by X and $k\langle X \rangle$ is a free algebra.

In this section, we introduce the CD-lemma for tensor product $k\langle X \rangle \otimes k\langle Y \rangle$ of free algebras, which is from L. A. Bokut, Y. Q. Chen and Y. S. Chen [23].

Let X and Y be linearly ordered sets, $S = \{yx = xy | x \in X, y \in Y\}$. Then, with the deg-lex order ($y > x$ for any $x \in X, y \in Y$) on $(X \cup Y)^*$, S is a Gröbner-Shirshov basis in $k\langle X \cup Y \rangle$. Then, the set

$$N = X^* Y^* = Irr(S) = \{u = u^X u^Y | u^X \in X^* \text{ and } u^Y \in Y^*\}$$

is the normal words of the tensor product of the free algebras

$$k\langle X \rangle \otimes k\langle Y \rangle = k\langle X \cup Y \mid S \rangle.$$

Let kN be a k -space spanned by N . For any $u = u^X u^Y, v = v^X v^Y \in N$, we define the multiplication of the normal words as follows

$$uv = u^X v^X u^Y v^Y \in N.$$

Then, kN clearly coincides with the tensor product $k\langle X \rangle \otimes k\langle Y \rangle$.

Now, we order the set N . For any $u = u^X u^Y, v = v^X v^Y \in N$,

$$u > v \Leftrightarrow |u| > |v| \text{ or } (|u| = |v| \text{ and } (u^X > v^X \text{ or } (u^X = v^X \text{ and } u^Y > v^Y))),$$

where $|u| = |u^X| + |u^Y|$ is the length of u . It is obvious that $>$ is a monomial order on N . Such an order is also called the deg-lex order on $N = X^* Y^*$.

Let f and g be monic polynomials of kN and $w = w^X w^Y \in N$. Then we have found 16 types of compositions of inclusion and intersection.

S is called a *Gröbner-Shirshov basis* in $kN = k\langle X \rangle \otimes k\langle Y \rangle$ if all compositions of elements in S are trivial modulo S .

Theorem 3.5 *Let $S \subseteq k\langle X \rangle \otimes k\langle Y \rangle$ with each $s \in S$ monic and “ $<$ ” the deg-lex order on $N = X^* Y^*$ as before. Then the following statements are equivalent:*

- (1) S is a Gröbner-Shirshov basis in $k\langle X \rangle \otimes k\langle Y \rangle$.
- (2) $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $a, b \in N, s \in S$.
- (3) $Irr(S) = \{w \in N \mid w \neq a\bar{s}b, a, b \in N, s \in S\}$ is a k -linear basis for the factor $k\langle X \cup Y \mid yx = xy, x \in X, y \in Y \rangle / Id(S)$.

4 Application of known CD-Lemmas

4.1 Schreier extensions of groups

Consider a Schreier extension of group

$$1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1.$$

Then we have Schreier's theorem (see [50]): A group G is a Schreier extension of A by B if and only if there exist a factor set $\{(b, b') \mid b, b' \in B\}$ of B in A and $\{b : A \rightarrow A, a \mapsto a^b \text{ is an automorphism}\}$ such that for any $b, b', b'' \in B, a \in A$,

$$(b, b')a^{bb'} = a^{[bb']}(b, b') \quad \text{and} \quad (b, b'b'')(b', b'') = (bb', b'')(b, b')^{b''},$$

where $[bb']$ is the product of elements b, b' in G .

M. Hall in his book [33] wrote down the following statement: “It is difficult to determine the identities [in A] leading to conditions for an extension”, where the group B is presented by generators and relations.

In a recent paper, Y. Q. Chen [21], by using Gröbner-Shirshov bases, the structure of Schreier extensions of groups is completely characterized and an algorithm is given to find conditions for any Schreier extension of a group A by B , where B is presented by a presentation. Therefore, the above problem of M. Hall is solved.

Let A, B be groups. By a factor set of B in A , we mean a subset of A which is related to the presentation of B , see below.

Let the group B be presented as semigroup by generators and relations: $B = \text{sgp}\langle Y | R \rangle$, where R is a Gröbner-Shirshov basis for B with the deg-lex order $<_B$ on Y^* . For the sake of convenience, we can assume that R is a minimal Gröbner-Shirshov basis in a sense that the leading monomials are not contained each other as subwords, in particular, they are pairwise different. Let G be as in the following Theorem 4.1, where $A_1 = A \setminus \{1\}$, $S = \{aa' = [aa'], v = h_v \cdot (v), ay = ya^y | v \in \Omega, a, a' \in A_1, y \in Y\}$, $\{(v) | v \in \Omega\} \subseteq A$ a factor set of B in A , $\psi_y : A \rightarrow A, a \mapsto a^y$ an automorphism.

We define a tower order on $(A_1 \cup Y)^*$ which extends the order $<_B$ on Y^* .

For $w_1 = w = v_1 c = d v_2, v_1, v_2 \in \Omega, c, d \in Y^*, \deg(v_1) + \deg(v_2) > \deg(w)$, we have,

$$f_{v_1} c - d f_{v_2} = d h_{v_2} - h_{v_1} c \equiv 0 \pmod{(R, w)}$$

It means that there exists a $z \in Y^*$ such that

$$h_{v_1} c \equiv d h_{v_2} \equiv z \pmod{(R, w)}$$

and thus, there exist $\xi_{(v_1, v_2)_w}(v), \zeta_{(v_1, v_2)_w}(v) \in A$ such that

$$g \equiv z(\xi_{(v_1, v_2)_w}(v) - \zeta_{(v_1, v_2)_w}(v)) \pmod{(S, w_1)} \quad (1)$$

where $\xi_{(v_1, v_2)_w}(v)$ and $\zeta_{(v_1, v_2)_w}(v)$ are functions of $\{(v) | v \in \Omega\}$, and $g = (v_1 - h_{v_1} \cdot (v_1), v_2 - h_{v_2} \cdot (v_2))_{w_1}$.

In fact, by the previous formulas, we have an algorithm to find the functions $\xi_{(v_1, v_2)_w}(v)$ and $\zeta_{(v_1, v_2)_w}(v)$.

Theorem 4.1 ([21]) *Let A, B be groups, $B = \text{sgp}\langle Y | R \rangle$, where $R = \{v - h_v | v \in \Omega\}$ is a minimal Gröbner-Shirshov basis for B and v the leading term of the polynomial $f_v = v - h_v \in R$. Let*

$$G = E(A, Y, a^y, (v)) = \text{sgp}\langle A_1 \cup Y | S \rangle$$

where $A_1 = A \setminus \{1\}$, $S = \{aa' = [aa'], v = h_v \cdot (v), ay = ya^y | v \in \Omega, a, a' \in A_1, y \in Y\}$, $\psi_y : A \rightarrow A, a \mapsto a^y$ an automorphism, $\{(v) | v \in \Omega\} \subseteq A$ a factor set of B in A with $(v) = 1$ if $f_v = y^\epsilon y^{-\epsilon} - 1, y \in Y, \epsilon = \pm 1$.

(i) *For the tower order, S is a Gröbner-Shirshov basis for G if and only if for any $v \in \Omega, a \in A$ and any composition $(f_{v_1}, f_{v_2})_w$ of R in $k\langle Y \rangle$,*

$$(v)a^v = a^{h_v}(v) \quad \text{and} \quad \xi_{(v_1, v_2)_w}(v) = \zeta_{(v_1, v_2)_w}(v) \quad (2)$$

hold in A , where $\xi_{(v_1, v_2)_w}(v)$ and $\zeta_{(v_1, v_2)_w}(v)$ are defined by (1). Moreover, if this is the case, G is a Schreier extension of A by B in a natural way.

(ii) A group C is a Schreier extension of A by B if and only if there exist $\{a^y | y \in Y, A \rightarrow A, a \mapsto a^y \text{ is an isomorphism}\}$ and a factor set $\{(v) | v \in \Omega\}$ of B in A with $(v) = 1$ if $f_v = y^\epsilon y^{-\epsilon} - 1, y \in Y, \epsilon = \pm 1$ such that (2) holds. Moreover, if this is the case, $C \cong G = E(A, Y, a^y, (v)) = \text{sgp}\langle A_1 \cup Y | S \rangle$.

Remark. In the above theorem, let the group $A = \text{gp}\langle X | R_A \rangle$ be also presented by generators and relations, where $R_A = \{u = f_u | u \in \Lambda\}$ is a Gröbner-Shirshov basis for A with the deg-lex order $<_A$ on X^* . Then, by replacing A_1 with X , $aa' = [aa']$ with R_A and x with a , the results hold.

As a corollary of the above theorem, in [21], by using the result in Y. Q. Chen and C. Y. Zhong in [27], a criteria in the case that B is HNN-extension is formulated.

Another solution to the M. Hall's problem can be referred in [2].

4.2 Extensions of algebras

In the paper of Y. Q. Chen [20], he gave the same kind of answer to an analogy of the M. Hall's problem in the above section for Schreier extensions of algebras.

Definition 4.2 Let k be a field, M, B, \mathcal{R} k -algebras (not necessarily with 1). Then \mathcal{R} is called an extension of M by B if $M^2 = 0$, where M is an ideal of \mathcal{R} and $\mathcal{R}/M \cong B$ as algebras. Such an extension is called a singular extension in [34].

The following classical result is known. Let M, B, \mathcal{R} be k -algebras with $M^2 = 0$. Then \mathcal{R} is an extension of M by B if and only if M is a B -bimodule and there exists a factor set $\{(b, b') | b, b' \in B\}$ of B in M such that for any $b, b', b'' \in B$,

$$b(b', b'') - (bb', b'') + (b, b'b'') - (b, b')b'' = 0.$$

In [20], by using Gröbner-Shirshov bases, the structure of extensions of algebras is completely characterized and an algorithm is given to find conditions for any extension of an algebra M by B , where B is presented by a presentation.

As results, by using this theorem, in [20], a characterization theorem of the extension of M by B is given, when B is a cyclic algebra, free commutative algebra, universal envelope of a Lie algebra, and Grassmann algebra, respectively.

4.3 Anti-commutative algebras

In the paper of L. A. Bokut, Y. Q. Chen and Y. Li [11], an application of Shirshov's CD-lemma for anti-commutative algebras (see [55]) is given. This application gives an anti-commutative Gröbner-Shirshov basis of a free Lie algebra.

Let $X = \{x_i | i \in I\}$ be a linear ordered set. Let X^{**} be the set of all non-associative words (u) in X . We assume that (u) is a bracketing of u . Then we define normal words $N = \{[u]\}$ and order them by using induction on the length $n = |[u]|$ of $[u]$:

If $n = 1$, then $[u] = x_i$ is a normal word. Define $x_i > x_j$ if $i > j$.

Let $N_{n-1} = \{[u] | [u] \text{ is a normal word and } |[u]| \leq n-1\}$, $n > 1$ and suppose that " $<$ " is a well order on N_{n-1} . Then

If $n > 1$ and $(u) = ((v)(w))$ is a word of length n , then (u) is a normal word, if and only if

both (v) and (w) are normal words, that is, $(v) = [v]$ and $(w) = [w]$, and $[v] > [w]$.

Let $[u], [v]$ be normal words of length $\leq n$. Then $[u] < [v]$, if and only if one of the following three cases holds:

$||[u]| < n, |[v]| < n$ and $[u] < [v]$.

$||[u]| < n$ and $|[v]| = n$.

If $||[u]| = |[v]| = n$, $[u] = [[u_1][u_2]]$ and $[v] = [[v_1][v_2]]$, then $[u_1] < [v_1]$ or $([u_1] = [v_1] \text{ and } [u_2] < [v_2])$.

It is clear that the order “ $<$ ” on N is a well order.

Let $AC(X)$ be a k -space spanned by normal words. Now, we define the product of normal words by the following way:

$$\widetilde{[u][v]} = \begin{cases} [[u][v]] & : [u] > [v] \\ -[[v][u]] & : [u] < [v] \\ 0 & : [u] = [v] \end{cases}$$

Then $AC(X)$ is the free anti-commutative algebra generated by X .

Let $S \subset AC(X)$ be a set of monic polynomials, $s \in S$ and $(u) \in X^{**}$. We define S -word $(u)_s$ by induction as a non-associative word in $X \cup S$ with only one occurrence of $s \in S$. An S -word $(u)_s$ is called a normal S -word if $(u)_{\bar{s}} = (a[\bar{s}]b)$ is a normal word.

There is only one kind of compositions that is inclusion one.

Theorem 4.3 ([55], [11]) *Let $S \subset AC(X)$ be a nonempty set of monic polynomials and the order “ $<$ ” as before. Then the following statements are equivalent:*

- (i) S is a Gröbner-Shirshov basis in $AC(X)$.
- (ii) $f \in Id(S) \Rightarrow [\bar{f}] = [a[\bar{s}]b]$ for some $s \in S$ and $a, b \in X^*$, where $[asb]$ is normal S -word.
- (iii) $Irr(S) = \{[u] \in N \mid [u] \neq [a[\bar{s}]b] \text{ } a, b \in X^*, s \in S \text{ and } [asb] \text{ is a normal } S\text{-word}\}$ is a basis of the algebra $AC(X|S)$.

By using this theorem, a Gröbner-Shirshov basis S in $AC(X)$ is given in [11] which shows that the Hall words in X forms a basis for the free Lie algebra $Lie(X)$, where $S = \{([u][v])[w] - ([u][w])[v] - [u]([v][w]) \mid [u] > [v] > [w] \text{ and } [u], [v], [w] \text{ are Hall words}\}$.

4.4 Akivis algebras

This section is from the paper of Y. Q. Chen and Y. Li [25].

An Akivis algebra is a vector space A over a field k endowed with a skew-symmetric bilinear product $[x, y]$ and a trilinear product (x, y, z) that satisfy the identity $[[x, y], z] + [[y, z], x] + [[z, x], y] = (x, y, z) + (z, x, y) + (y, z, x) - (x, z, y) - (y, x, z) - (z, y, x)$. For any (non-associative) algebra B , one may obtain an Akivis algebra $Ak(B)$ by considering in B the usual commutator $[x, y] = xy - yx$ and associator $(x, y, z) = (xy)z - x(yz)$.

The CD-lemma for non-associative algebras is invented by Shirshov in [55]. By applying this lemma, in [25], a Gröbner-Shirshov basis in $AC(X)$ is given for the universal enveloping algebra of an Akivis algebra which gives another proof of I. P. Shestakov's result (see [51]) that any Akivis algebra is linear. An Akivis algebra A is linear if A can be embedded in some non-associative algebra B with above operations.

Theorem 4.4 ([25]) *Let $(A, +, [,], (, ,))$ be an Akivis algebra with a linearly ordered basis $\{e_i \mid i \in I\}$. Let*

$$[e_i, e_j] = \sum_m \alpha_{ij}^m e_m, \quad (e_i, e_j, e_k) = \sum_n \beta_{ijk}^n e_n,$$

where $\alpha_{ij}^m, \beta_{ijk}^n \in k$. We denote $\sum_m \alpha_{ij}^m e_m$ and $\sum_n \beta_{ijk}^n e_n$ by $\{e_i e_j\}$ and $\{e_i e_j e_k\}$, respectively.

Let

$$U(A) = M(\{e_i\}_I \mid e_i e_j - e_j e_i = \{e_i e_j\}, (e_i e_j) e_k - e_i (e_j e_k) = \{e_i e_j e_k\}, i, j, k \in I)$$

be the universal enveloping algebra of A . Let

$$S = \{e_i e_j - e_j e_i - \{e_i e_j\} \mid (i > j), (e_i e_j) e_k - e_i (e_j e_k) - \{e_i e_j e_k\} \mid (i, j, k \in I), e_i (e_j e_k) - e_j (e_i e_k) - \{e_i e_j\} e_k - \{e_j e_i e_k\} + \{e_i e_j e_k\} \mid (i > j, k \geq j)\}.$$

Then

- (i) S is a Gröbner-Shirshov basis for $U(A)$,
- (ii) A can be embedded into the universal enveloping algebra $U(A)$.

4.5 Some one-relator groups

This section contains the results of Y. Q. Chen and C. Y. Zhong in [28].

It is well known the Magnus algorithm for a solution of the word problem for any one-relator group.

Using the Magnus rewriting procedure (see R. C. Lyndon and P. E. Schupp [43]), one may embed any one-relator group into a tower of HNN-extensions. For towers of HNN-extensions of groups, L. Bokut (see [13]) developed a method of groups with the standard normal forms in 1965. Actually, for these groups, Gröbner-Shirshov bases are also “standard” in a sense, so we may speak about “groups with the standard Gröbner-Shirshov bases”.

By the way, we give here a story about applications of groups with the standard normal forms. At our seminar, we have studied the following interesting paper

K. Kalorkoti, *Turing degree and the word and conjugacy problem for finitely presented groups*, Internet.

Actually, it is a part of his Thesis in London University, 1979.

C. Y. Zhong was the speaker for almost two months at the seminar and she pointed out that K. Kalorkoti used successfully the method of L. A. Bokut on groups with the standard normal forms.

As a result of this study, we suggested K. Kalorkoti to publish his paper in the SEA Bull. Math. (see [35]).

Now, we back to one-relator groups. There is a chance that any tower of HNN-extensions produced by the Magnus method has the standard Gröbner-Shirshov basis and the standard normal form. In particular, it would give another algorithm for the word problem for any one-relator group.

The problem is that the Magnus embedding is not easy to observe and write down explicitly. Hence, we need to go step by step. Any one-relator group can be effectively embedded into one-relator group with two generators

$$gp\langle X|r \rangle \hookrightarrow G = gp\langle x, y | x^{n_1}y^{m_1} \cdots x^{n_k}y^{m_k} = 1 \rangle,$$

where $x_i \mapsto x^{-q}y^l$ (W. Magnus, see also [43]), $n_i, m_i \neq 0, k \geq 0$. We call k the depth of G .

Theorem 4.5 ([28]) *Any two-generator one-relator group of the depth 3 is effectively Magnus embeddable into a tower of HNN-extensions, which is a group with the effective standard Gröbner-Shirshov basis and effective standard normal form.*

4.6 The Chinese monoid

The Chinese monoid $CH(X)$ on a well ordered set X has the following defining relations:

$$cba = bca = cab, \quad c \geq b \geq a, \quad a, b, c \in X.$$

A fundamental paper on the Chinese monoid has been published in 2001 [19].

In the paper of Y. Q. Chen and J. J. Qiu [26], a Gröbner-Shirshov basis for the Chinese monoid is found.

Theorem 4.6 ([26]) *Let $S = \{x_i x_j x_k - x_j x_i x_k, x_i x_k x_j - x_j x_i x_k, x_i x_j x_j - x_j x_i x_j, x_i x_i x_j - x_i x_j x_i, x_i x_j x_i x_k - x_i x_k x_i x_j, x_i, x_j, x_k \in X, i > j > k\}$. Then*

(i) *$sgp\langle X|T \rangle = sgp\langle X|S \rangle$ and with deg-lex order, S is a Gröbner-Shirshov basis of the Chinese monoid $CH(X)$.*

(ii) *The $Irr(S)$ is the set which consists of words on X of the form $u_n = w_1 w_2 \cdots w_n, n \geq 0$, where*

$$\begin{aligned} w_1 &= x_1^{t_{11}} \\ w_2 &= (x_2 x_1)^{t_{21}} x_2^{t_{22}} \\ w_3 &= (x_3 x_1)^{t_{31}} (x_3 x_2)^{t_{32}} x_3^{t_{33}} \\ &\quad \dots \\ w_n &= (x_n x_1)^{t_{n1}} (x_n x_2)^{t_{n2}} \cdots (x_n x_{n-1})^{t_{n(n-1)}} x_n^{t_{nn}} \end{aligned}$$

with $x_i \in X, x_1 < x_2 < \cdots < x_n$ and all exponents are non-negative integers.

Then $Irr(S)$ coincides with the set of staircase words of the paper by J. Cassaigne et al. [19]. Also, the insertion algorithm of J. Cassaigne et al. [19] coincides with the elimination of leading words algorithm.

4.7 Markov and Artin normal form theorem for braid groups

In the paper of L. A. Bokut, V. V. Chaynikov and K. P. Shum [8], the authors present the classical results of Artin-Markov on braid groups by using the Gröbner-Shirshov bases. As an application, one can reobtain the normal form of Artin-Makov-Ivanovskiy as an easy corollary.

5 Expository papers

5.1 Gröbner and Gröbner-Shirshov bases: an elementary approach

There is an elementary approach in L. A. Bokut and K.P. Shum [15] to Gröbner-Shirshov bases theory with quite a few examples, including the example for Lie algebras.

5.2 Shirshov's CD-lemma for Lie algebras

What is now called the Gröbner-Shirshov method for Lie algebras originally invented by A. I. Shirshov in 1962 ([56]). Actually, that paper based on his paper [53] where Shirshov invented a new linear basis for a free Lie algebra now called the Lyndon-Shirshov basis (it was defined independently in the paper [24] in the same year). Remarkably, Lyndon-Shirshov basis is a particular case of the series of bases of a free Lie algebra invented by A. I. Shirshov in his Candidate of Doctor of Science Thesis (Moscow State University, 1953, advisor was A. G. Kurosh) and published in 1962 ([54]) (cf. [49] where these bases are called Hall Bases). We now cite the Zbl review by P. M. Cohn [31] of the paper [53]: “The author varies the usual construction of basis commutators in Lie rings by ordering words lexicographically and not by length. This is used to give a very short proof of the theorem (Magnus [44], Witt [58]) that the Lie algebra obtained from a free associative algebra is free. Secondly he derives Friedrich's criterion (this Zbl 52,45) for Lie elements. As the third application he proves that every Lie algebra L can be embedded in a Lie algebra M such that in M any subalgebra of countable dimension is contained in a 2-generated subalgebra.” We would like to add that it was a beginning of Gröbner-Shirshov bases theory for Lie and associative algebras. Lemma 4 of the paper, on special bracketing of a regular (Lyndon-Shirshov) associative word with a fix regular subword, leads to the algorithm of elimination of the leading word of one Lie polynomial in other Lie polynomial, i.e., to the reduction procedure that is very familiar in the cases of associative and associative-commutative polynomials. Also this Lemma 4 leads to the crucial notion of composition of two Lie polynomials that will be defined lately in the paper [56]. Last but not least, Shirshov [52] proved the following result for connections of some ideals of free Lie and free associative algebras.

Let $Lie(a, b)$ be the Lie algebra of Lie polynomials of $k\langle a, b \rangle$ (it is the free Lie algebra over a set $\{a, b\}$ and a field k). Let $J = J([a^2b^k ab] = [[a[[ab] \cdots b]][ab], k \geq 1)$ be the

Lie ideal of $Lie(a, b)$ generated by $\{[a^2b^k ab], k \geq 1\}$ and I the associative ideal of $k\langle a, b \rangle$ generated by J . Then, $I \cap Lie(a, b) = J$. The proof is dealing with leading monomials of Lie and associative polynomials. This result and its proof are the real beginning of Gröbner-Shirshov bases theory for Lie and associative algebras.

As for the paper [56] itself, it is a fully pioneer paper in the subject. He defines a notion of the composition $(f, g)_w$ of two Lie (associative) polynomials relative to an associative word w (it was called lately by S -polynomial for commutative polynomials by B. Buchberger [17] and [18]). It leads to the algorithm for construction of a Gröbner-Shirshov basis ($GSB(S)$) of Lie (associative) ideal generated by some set S : to joint to S all nontrivial compositions and to eliminate leading monomials of one polynomial of S in others. Shirshov proves the lemma, now known as the Composition, or Composition-Diamond lemma, that if $f \in Id_{Lie}(S)$, then \bar{f} , the leading associative word of f , has a form $\bar{f} = u\bar{s}v$, where $s \in GSB(S)$, $u, v \in X^*$. Several years later the first author formulated this lemma in the modern form (see [6]). Let S be a set of Lie polynomials that is complete under composition (i.e., any composition of polynomials of S is trivial; on the other word, S is a Gröbner-Shirshov basis). Then if $f \in Id_{Lie}(S)$, then $\bar{f} = u\bar{s}v$, where $s \in S$, $u, v \in X^*$. Of course, from Shirshov's Composition-Diamond lemma it follows that the set $Irr(S)$ of S -irreducible Lyndon-Shirshov words constitutes a linear basis of the quotient algebra $Lie(X)/Id(S)$. The converse is also true.

Lately explicitly Shirshov's Composition-Diamond lemma for associative algebra was formulated by L. A. Bokut [7] in 1976 and G. Bergman [3] in 1978.

The paper by L. A. Bokut and Y. Q. Chen [9] contains a comprehensive proof of the Shirshov's Composition-Diamond lemma for Lie algebras. We follow Shirshov's ideas of his papers [52], [53], [54] and [56]. In particular, we prove all necessary properties of both associative (see also [42]) and non-associative Lyndon-Shirshov words by using the Shirshov's elimination in [52] (that is, the so called Lazard elimination in [41] and [49]).

5.3 CD-lemma for modules

Composition-Diamond lemma for modules was first formulated and proved by S.-J. Kang and K.-H. Lee in [36] and [37]. According to their approach, a Gröbner-Shirshov basis of a cyclic module M over an algebra A is a pair (S, T) , where S is the set of the defining relations of A , $A = k\langle X|S \rangle$, and T is the defining relations for the A -module ${}_A M = {}_A M(e|T)$. Then Kang-Lee's Lemma says that (S, T) is a Gröbner-Shirshov pair for the A -module ${}_A M = {}_A M(e|T)$ if S is a Gröbner-Shirshov basis of A and T is closed under the right-justified composition with respect to S , and for $f \in S$, $g \in T$, such that $(f, g)_w$ is defined and $(f, g)_w \equiv 0 \mod(S, T, w)$. A composition $(f, g)_w$ is called right-justified if $w = \bar{f} = a\bar{g}$ for some $a \in X^*$.

Some years later, E. S. Chibrikov [29] suggested a new Composition-Diamond lemma for modules that treat any module as a factor module of "double-free" module.

Let X, Y be sets and $mod_{k\langle X \rangle}\langle Y \rangle$ a free left $k\langle X \rangle$ -module with the basis Y . Then $mod_{k\langle X \rangle}\langle Y \rangle = \bigoplus_{y \in Y} k\langle X \rangle y$ is called a "double-free" module.

Suppose that $<$ is a monomial order on X^* , $<$ a well order on Y and $X^*Y = \{uy|u \in X^*, y \in Y\}$. We define an order " \prec " on X^*Y : for any $w_1 = u_1y_1, w_2 = u_2y_2 \in X^*Y$,

$$w_1 \prec w_2 \Leftrightarrow u_1 < u_2 \quad \text{or} \quad u_1 = u_2, y_1 < y_2 \quad (3)$$

Let $S \subset \text{mod}_{k\langle X \rangle} \langle Y \rangle$ with each $s \in S$ monic. We define a composition $(f, g)_w = f - ag$, where $w = \bar{f} = a\bar{g}$, $a \in X^*$, $f, g \in \text{mod}_{k\langle X \rangle} \langle Y \rangle$ are monic.

If $(f, g)_w = f - ag = \sum \alpha_i a_i s_i$, where $\alpha_i \in k$, $a_i \in X^*$, $s_i \in S$ and $a_i \bar{s}_i \prec w$, then this composition is called trivial modulo (S, w) and is denoted by $(f, g)_w \equiv 0 \text{ mod}(S, w)$.

Definition 5.1 ([29]) *Let $S \subset \text{mod}_{k\langle X \rangle} \langle Y \rangle$ be a non-empty set with each $s \in S$ monic. Let the order “ \prec ” be as before. Then we call S a Gröbner-Shirshov basis in the module $\text{mod}_{k\langle X \rangle} \langle Y \rangle$ if all the compositions of polynomials in S are trivial modulo S .*

Lemma 5.2 ([29], Composition-Diamond lemma for “double-free” modules) *Let $S \subset \text{mod}_{k\langle X \rangle} \langle Y \rangle$ be a non-empty set with each $s \in S$ monic and “ \prec ” the order on X^*Y as before. Then the following statements are equivalent:*

- (i) S is a Gröbner-Shirshov basis in $\text{mod}_{k\langle X \rangle} \langle Y \rangle$.
- (ii) $f \in k\langle X \rangle S \Rightarrow \bar{f} = a\bar{s}$ for some $a \in X^*$, $s \in S$.
- (iii) $\text{Irr}(S) = \{w \in X^*Y \mid w \neq a\bar{s}, a \in X^*, s \in S\}$ is a k -linear basis for the factor $\text{mod}_{k\langle X \rangle} \langle Y \mid S \rangle = \text{mod}_{k\langle X \rangle} \langle Y \rangle / k\langle X \rangle S$.

As applications of Lemma 5.2, Y. Q. Chen, Y. S. Chen and C. Y. Zhong [22], found the Gröbner-Shirshov bases for highest weight modules over Lie algebra sl_2 , Verma modules over Kac-Moody algebras, Verma modules over Lie algebras of coefficients of free conformal algebras and the universal enveloping modules for Sabinin algebras. (The last modules are defined in [48]).

References

- [1] M. Aymon and P.-P. Grivel, Un theoreme de Poincare-Birkhoff-Witt pour les algebres de Leibniz, *Comm. Algebra*, **31**, 527-544(2003).
- [2] Y. G. Baik, J. Harlander and S. J. Pride, The geometry of group extensions, *J. Group Theory*, **1**, 4, 395-416(1998).
- [3] G. M. Bergman, The diamond lemma for ring theory, *Adv. in Math.*, **29**, 178-218(1978).
- [4] L. A. Bokut, On one property of the Boone groups, *Algebra i Logika*, **5**, 5, 5-23(1966), **II**, **6**, 1, 25-38(1967).
- [5] L. A. Bokut, Groups of functions of multiplication semigroups of certain rings I, II, III, IV, *Sibir. Math. J.*, **10**, 2, 4, 4, 5, 246-286, 744-199, 800-819, 965-1005(1969).
- [6] L. A. Bokut, Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.*, **36**, 1173-1219(1972).
- [7] L. A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, **15**, 117-142(1976).

- [8] L. A. Bokut, V. V. Chaynikov and K. P. Shum, Markov and Artin normal form theorem for braid groups, *Comm. Algebra*, **35**, 2105-2115(2007).
- [9] L. A. Bokut and Yuqun Chen, Gröbner-Shirshov bases for Lie algebras: after A.I. Shirshov, *SEA Bull Math.*, **31**, 1057-1076(2007).
- [10] L. A. Bokut, Yuqun Chen and Cihua Liu, Gröbner-Shirshov bases for dialgebras, submitted.
- [11] L. A. Bokut, Yuqun Chen and Yu Li, Anti-commutative Gröbner-Shirshov bases of a free Lie algebra, submitted.
- [12] L. A. Bokut, Y. Fong and W. F. Ke, Composition Diamond Lemma for associative conformal algebras, *J. Algebra*, **272**, 739-774(2004).
- [13] L. A. Bokut and G. Kukin, Undecidable algrithmic problems for semigroup, *Algebra, Topology and Geometry*, **25**, 3-66. VINITI, Moscow, 1987.
- [14] L. A. Bokut, and K. P. Shum, Relative Gröbner-Shirshov bases for algebras and groups, *Algebra and Analisis*, to appear.
- [15] L. A. Bokut and K. P. Shum, Gröbner and Gröbner-Shirshov bases in algebra: an elementary approach, *SEA Bull. Math.*, **29**, 227-252(2005).
- [16] N. Bourbaki, Lie Groups and Lie Algebras, Chapter 4-6, Elements of Mathematics, Springer-Verlag, 2002.
- [17] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal [in German], Ph.D. thesis, University of Innsbruck, Austria, 1965.
- [18] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations [in German], *Aequationes Math.*, **4**, 374-383(1970).
- [19] J. Cassaigne, M. Espie, D. Krob, J. C. Novelli and F. Hivert, The Chinese Monoid, *International Journal of Algebra and Computation*, **11**, 3, 301-334(2001).
- [20] Yuqun Chen, Gröbner-Shirshov basis for Schreier extensions of groups, *Comm. Algebra*, to appear. DOI: 10.1080/00927870701869899
- [21] Yuqun Chen, Gröbner-Shirshov basis for extensions of algebras, *Algebra Colloq.*, to appear.
- [22] Yuqun Chen, Yongshan Chen and Chanyan Zhong, Composition-Diamond Lemma for Modules, submitted.
- [23] L. A. Bokut, Yuqun Chen and Yongshan Chen, Composition-Diamond lemma for tensor product of free algebras, submitted.
- [24] K. T. Chen, R. H. Fox, and R. C. Lyndon, Free differential calculus, IV: the quotient groups of the lower central series. *Annals of Mathematics*, **68**, 81-95(1958).

- [25] Yuqun Chen and Yu Li, An application of non-associative Composition-Diamond lemma, submitted.
- [26] Yuqun Chen and Jianjun Qiu, Gröbner-Shirshov Basis for the Chinese Monoid, submitted.
- [27] Yuqun Chen and Chanyan Zhong, Gröbner-Shirshov basis for HNN extensions of groups and for the alternative group, *Comm. Algebra*, **36**, 1, 94-103(2008).
- [28] Yuqun Chen and Chanyan Zhong, Gröbner-Shirshov basis for some one-relator groups, submitted.
- [29] E. S. Chibrikov, On free Lie conformal algebras, *Vestnik Novosibirsk State University*, **4**, 1, 65-83(2004).
- [30] E. S. Chibrikov, A right normed basis of free Lie algebras and Lyndon-Shirshov words. *J. Algebra*, **302**, 593-612(2006).
- [31] P. M. Cohn, review Zbl 0080.25503 Shirshov, A. I. Über freie Liesche Ringe. (Russian) *Mat. Sb., N. Ser.*, **45**(87), 113-122(1958).
- [32] G. Duchamp and D. Krob, *Plactic-growth-like monoids*, in Words, Languages and Combinatorics II, Kyoto, Japan, 25-28 August 1992, eds. M Jürgensen, World Scientific, 1994, 124-142.
- [33] Marshall Hall, Jr., The Theory of Groups, The Macmillan Company, 1959.
- [34] G. Hochschild, On the cohomology groups of an associative algebra, *Ann. Math.*, **46**, 58-67(1945).
- [35] K. Kalorkoti, Turing degrees and the word conjugacy problems for finitely presented groups, *SEA. Bull. Math.*, **30**, 855-888(2006).
- [36] S.-J. Kang, K.-H. Lee, Gröbner-Shirshov bases for representation theory, *J. Korean Math. Soc.*, **37**, 55-72(2000).
- [37] S.-J. Kang, K.-H. Lee, Gröbner-Shirshov bases for irreducible sl_{n+1} -modules, *J. Algebra*, **232**, 1-20(2000).
- [38] D. E. Knuth and P. B. Bendix, Simple word problems in universal algebras, *Comput. Probl. abstract Algebra, Proc. Conf. Oxford 1967*, 263-297 (1970).
- [39] P. Kolesnikov, Conformal representations of Leibniz algebras, arXiv:math/0611501.
- [40] J.-L. Loday, Algebras with two associative operations (dialgebras), *C. R. Acad. Sci. Paris*, **321**, 141-146(1995).
- [41] M. Lothaire, "Combinatorics on Words", Encyclopedia of Mathematics, Vol. 17, Addison-Wesley, 1983; reprinted M. Lothaire, "Combinatorics on Words", Cambridge Mathematical Library, Cambridge University Press, 1997.
- [42] R. C. Lyndon, On the Burnside problem, *Trans. Amer. Math. Soc.*, **77**, 202-215(1954).

- [43] R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Springer-Verlag, 1977.
- [44] W. Magnus, Über Beziehungen zwischen höheren Kommutatoren, *J. Reine Angew. Math.*, **177**, 105-115(1937).
- [45] A. A. Mikhalev, The junction lemma and the equality problem for color Lie superalgebras, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, **5**, 88-91(1989). English translation: *Moscow Univ. Math. Bull.*, **44**, 87-90(1989).
- [46] A. A. Mikhalev and E. A. Vasilieva, Standard bases of ideals of free supercommutative polynomial algebras (ε -Gröbner bases). Proc. Second International Taiwan-Moscow Algebra Workshop, Springer-Verlag, in press.
- [47] A. A. Mikhalev and A. A. Zolotykh, Standard Gröbner-Shirshov bases of free algebras over rings, I. Free associative algebras, preprint.
- [48] J. M. Perez-Izquierdo, Algebras, hyperalgebras, nonassociative bialgebras and loops, *Advances in Mathematics*, **208**, 834-876(2007).
- [49] C. Reutenauer, Free Lie algebras. Oxford Science Publications, 1993.
- [50] Otto Schreier, Über die Erweiterung von Gruppen I, *Monats. für Math. u. Phys.*, **34**, 165-180(1926).
- [51] I. P. Shestakov, Every Akivis algebra is linear, *Geometriae Dedicata*, **77**, 215-223(1999).
- [52] A. I. Shirshov, Subalgebras of free Lie algebras, *Mat. Sb.*, **33**, 441-452(1953) (in Russian).
- [53] A. I. Shirshov, On free Lie rings, *Mat. Sbornik*, **45**(87), 113-122(1958).
- [54] A. I. Shirshov, Bases for free Lie algebras, *Algebra Logic*, **1**, 1, 14-19(1962).
- [55] A. I. Shirshov, Some algorithmic problems for ε -algebras, *Sibirsk. Mat. Z.* **3**, 1, 132-137(1962).
- [56] A. I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, **3**, 292-296(1962) (in Russian); English translation in SIGSAM Bull., **33**(2), 3-6(1999).
- [57] T. Stokes, Gröbner bases in exterior algebra, *J. Automated Reasoning*, **6**, 233-250(1990).
- [58] E. Witt, Treue Darstellungen Liescher Ringe, *J. Reine Angew. Math.*, **177**, 152-160(1937).